Nature of perturbation theory in spin glasses

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The high-order behavior of the perturbation expansion in the cubic replica field theory of spin glasses in the paramagnetic phase has been investigated. The study starts with the zero-dimensional version of the replica field theory and this is shown to be equivalent to the problem of finding finite size corrections in a modified spherical spin glass near the critical temperature. We find that the high-order behavior of the perturbation series is described, to leading order, by coefficients of alternating signs (suggesting that the cubic field theory is well-defined) but that there are also subdominant terms with a complicated dependence of their sign on the order. Our results are then extended to the d-dimensional field theory and in particular used to determine the high-order behavior of the terms in the expansion of the critical exponents in a power series in $\epsilon = 6 - d$. We have also corrected errors in the existing ϵ expansions at third order.

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I. INTRODUCTION

The theory of spin glasses in finite dimensional systems has traditionally been approached by the loop expansion around Parisi's mean-field replica symmetry breaking solution [1]. However, the picture of spin glasses which emerges from this perturbative approach is quite different to the one arising in droplet theory [2]. The motivation for the present paper was to investigate the possibility that perturbation theory in spin glasses might fail for some reason (for example, non-perturbative terms like "droplets" might dominate their free energy, at least in the low-temperature phase). We have started the programme with a study of the nature of the perturbation expansion in the high-temperature or paramagnetic phase, and postpone to another paper the discussion of the low-temperature phase.

In general the nature of perturbation expansions in disordered system is far from trivial. This is in contrast to the perturbation theory in pure systems, where, for example, the Borel summability of the series leads to an accurate evaluation of critical exponents [3]. In the study of the perturbation expansion for disordered ferromagnets, Bray et al. [4] found that even in zero dimensions, the high-order behavior of the perturbation expansion was surprisingly rich. The high-order expansion coefficients are sums of two kinds of terms: one type has an unusual cosine-like oscillation with increasing periodicity and the second type has a simple alternation in sign which dominates for small disorder. This unusual behavior has been further studied in Refs. [5], [6], with the final conclusion that the series is still summable, but that the simple Borel procedure needs to be modified to deal successfully with long series. We shall find that the perturbation expansion for spin glasses has remarkably similar features to those of the disordered ferromagnet.

Our investigation of the nature of the perturbation expansions in the high-temperature phase of spin glasses starts by looking at the perturbation expansion of the zero-dimensional spin glass problem (which will be referred to as the "toy" problem). The zero-dimensional field theory is the key to the analysis of the d-dimensional field theory as the extension to the d-dimensional field theory and critical exponents is a relatively straightforward extension of the toy problem [7]. Apart from being simple integrals, the zero-dimensional toy field theory has the advantage of allowing an analysis without replicas. In this paper this is achieved by mapping the problem to that of critical finite-size scaling in a modified version of the spherical spin glass [8] and this mapping allowed us to find the high-order behavior without the use of replicas. However, to obtain the high-order behavior of the perturbation expansion for the d-dimensional field theory requires the use of replicas and we found that this needed the use of a non-trivial replica symmetry breaking scheme in the toy model in order to get results consistent with our mapping to the spherical model. Our chief result is that the perturbation theory is well-defined and the dominant high-order terms in the perturbation expansion have coefficients of alternating signs. However, the perturbation series of the zero-dimensional spin glass field theory is not Borel summable in a straightforward way due to the presence of subdominant terms.

The paper is organized as follows. In the next section, we consider the cubic replica field theory of spin glasses and obtain the first few expansion coefficients of the zero-dimensional toy problem by explicitly evaluating the Feynman diagrams. In Sec. III, we show the equivalence of the zero-dimensional field theory and the critical finite-size scaling of the modified spherical model. Using this mapping we obtain the high order behavior of the perturbation expansion for the toy problem in Sec. IV. In Sec. V, we consider the toy problem using replicas. This is generalized to the problem of the high order terms in the ϵ expansion in Sec. VI. We conclude with a discussion in Sec. VII.

II. REPLICA FIELD THEORY OF SPIN GLASSES

The replica field theory of spin glasses, (see Ref. [9] for a review), starts from the Hamiltonian density

$$\mathcal{H} = \frac{1}{4}\sum_{\alpha,\beta}(\nabla q_{\alpha\beta})^2 + \frac{\tau}{4}\sum_{\alpha,\beta} q_{\alpha\beta}^2 - \frac{w}{6}\sum_{\alpha,\beta,\gamma} q_{\alpha\beta}q_{\beta\gamma}q_{\gamma\alpha} \quad (1)$$
$$- y\Big(\frac{1}{12}\sum_{\alpha,\beta} q_{\alpha\beta}^4 + \frac{1}{8}\sum_{\alpha,\beta,\gamma,\delta} q_{\alpha\beta}q_{\beta\gamma}q_{\gamma\delta}q_{\delta\alpha} - \frac{1}{4}\sum_{\alpha,\beta,\gamma} q_{\alpha\beta}^2 q_{\alpha\gamma}^2\Big).$$

As usual the field components $q_{\alpha\beta}$ ($\alpha \neq \beta$ and $q_{\alpha\beta} = q_{\beta\alpha}$) take all real values, and the indices such as α take the values $1, 2, 3, \ldots, n$. In the limit when n goes to zero, such a field theory is thought to capture the physics of finite dimensional spin glasses. The quartic terms work as stabilizing terms, but for $d < 6$ are irrelevant variables which we shall drop. One question which we shall ask is whether the resulting cubic field theory is well-defined in the limit $n \to 0$. It is possible this approach may not be valid as cubic theories have Hamiltonians which are not bounded below [7]. However, there are examples of field theories existing where unphysical limits, such as the number of field components n is taken to zero, which saves these apparently unphysical theories [7, 10]. This seems also to be the case for spin glasses as our work shows that the coefficients in the perturbation expansion in w alternate in sign, which is an indication that the field theory remains well-defined in the limit when n goes to zero.

The partition function of the zero-dimensional spin glass field theory is given by

$$Z = \int \prod_{\alpha<\beta}\left(\frac{dq_{\alpha\beta}}{\sqrt{2\pi}}\right)\exp\Big[-\frac{\tau}{4}\sum_{\alpha,\beta} q_{\alpha\beta}^2 + \frac{w}{6}\sum_{\alpha,\beta,\gamma} q_{\alpha\beta}q_{\beta\gamma}q_{\gamma\alpha}\Big]. \quad (2)$$

The perturbation expansion is well-defined irrespective of whether the integral of Eq. (2) actually exists. The perturbation expansion in w yields a series

$$Z(g^2) = \tau^{-\frac{n(n-1)}{4}}\left[1 + \sum_{K=1}^{\infty} A_K g^{2K}\right], \quad (3)$$

where we take $g^2 = w^2/(\tau/2)^3$ as the expansion parameter of the problem. The use of $\tau/2$ instead of τ is for later convenience. The series expansion of the corresponding free energy is given by

$$\beta F(g^2) = \frac{n(n-1)}{4}\ln\tau - \sum_{K=1}^{\infty} B_K g^{2K}. \quad (4)$$

Although this zero-dimensional theory is nothing but a multiple integral, it is not an easy task to calculate the expansion coefficients A_K or B_K directly by expanding the exponential in (2), because of the complicated structure of the internal summations. In fact, even using a symbolic manipulation program on a computer, we find it very difficult to get the expansion coefficients higher order than the first couple of terms. Fortunately, in the study of a cubic field theory similar to ours – the percolation problem [11] – different types of the internal contractions occurring in the theory were classified diagrammatically, and all the relevant diagrams were given up to $O(g^8)$. To this order there are five different internal contraction types. (See the diagrams denoted by α, β, γ, δ and λ in Figs. 1 and 2.) These were translated into the spin glass problem in Ref. [12]. In Figs. 1 and 2, all the diagrams to this order contributing to the renormalized propagator and vertex are listed along with their contributions. Note that, in the zero-dimensional field theory, each contribution is just given by the product of these contraction factors.

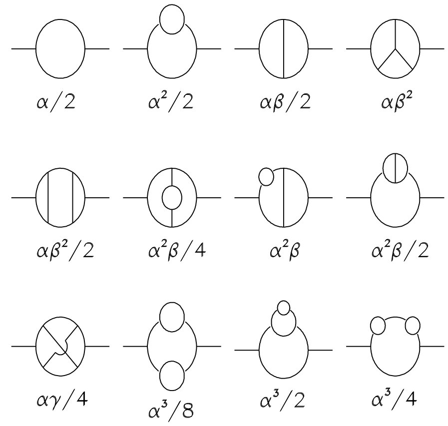

FIG. 1: Feynman diagrams to $O(g^8)$ contributing to the renormalized propagator with their corresponding contributions.

The expansion coefficient B_K of the free energy can be obtained by noting that $-\partial(\beta F)/\partial w$ is just the renormalized three-point function, which can be written in terms of the renormalized propagator and vertex. Collecting the contribution from each diagram and using the results of the internal contractions, we calculate the perturbation expansion coefficient B_K up to four-loop order $(0(g^8))$. We find that $B_K = \frac{1}{6}n(n-1)(n-2)f_K/8^K$, where $f_1 = 1/2$, $f_2 = n-2$, $f_3 = 4n^2 - 31n/2 + 44/3$ and $f_4 = 22n^3 - 123n^2 + 229n - 148$. Therefore, the free energy for the toy spin glass problem, $\lim_{n\to 0} F/n$, is given by a power series in g^2 with coefficients of alternating signs up to $O(g^8)$.

In the course of investigation, we have discovered errors in two (δ and λ in Figs. 1 and 2) of the five types of contraction reported in Ref. [12]. The correct results we obtain are $\delta = n^3 - 9n^2 + 54n - 104$ and $\lambda = 5n^2 - 14n$,

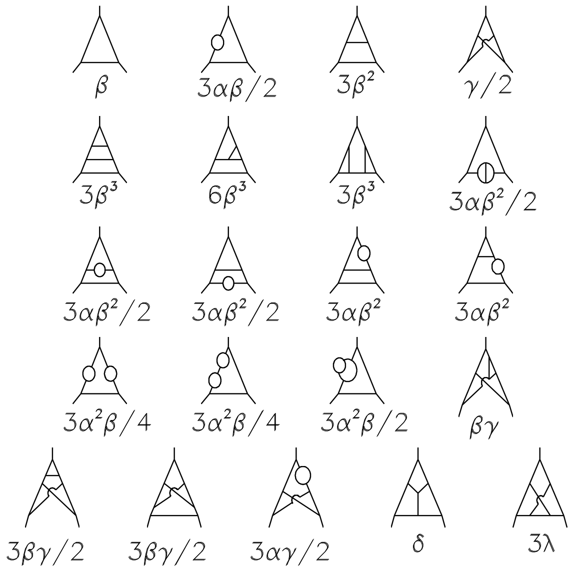

FIG. 2: Feynman diagrams to O(g^8) contributing to the renormalized vertex with their corresponding contributions.

while the results in Ref. [12] were $\delta = n^3 - 3n^2 + 38n - 94$ and $\lambda = 5n^2 - 2n - 12$. The latter can easily be shown to be incompatible with the simple $n = 3$ case. These corrections will change the O(ϵ^3) terms in the critical exponents in $d = 6 - \epsilon$ dimensions. From the explicit expressions for the critical exponents η and ν obtained in Ref. [11], we calculate the correct form of the ϵ expansion to third order to be

$$\eta = -0.3333\epsilon + 1.2593\epsilon^2 + 0.7637\epsilon^3 \tag{5}$$
$$\nu^{-1} - 2 + \eta = -2\epsilon + 9.2778\epsilon^2 - 6.4044\epsilon^3. \tag{6}$$

The corrected series for $\nu^{-1}-2+\eta$ shows an oscillation in signs in contrast to the one in Ref. [12] where the $O(\epsilon^3)$ term was positive.

III. MAPPING OF THE TOY PROBLEM ONTO A MODIFIED SPHERICAL MODEL

As mentioned in the Introduction, we study the zero-dimensional field theory by mapping it onto a modified version of the spherical spin glass model. The Hamiltonian of the spherical model is

$$H_{\rm sp} = -\frac{1}{2}\sum_{i,j} J_{ij} S_i S_j, \tag{7}$$

with the spherical constraint $\sum_i S_i^2 = N$ among the N spins. Unlike the conventional spherical spin glass model [8], we take $J_{ii} \neq 0$ in addition to the infinite-ranged interactions $J_{ij} = J_{ji}$ $(i \neq j)$. They are chosen from Gaussian distributions

$$P(J_{ij}) \sim e^{-\frac{N}{4}{\rm tr}\mathbf{J}^2} = \prod_i e^{-\frac{N}{4}J_{ii}^2} \prod_{i<j} e^{-\frac{N}{2}J_{ij}^2}. \tag{8}$$

The presence of the diagonal interaction does not make any difference in the $N \to \infty$ limit. In the following, however, we consider finite size corrections in this model. The partition function can be written as

$$\begin{aligned} Z_{\rm sp} &= \frac{\beta}{2}\int_{-\infty}^{\infty}\prod_i dS_i \int_{-i\infty}^{i\infty}\frac{dz}{2\pi i} \\ &\quad\times\ \exp\Big[\frac{\beta}{2}\big(Nz - z\sum_i S_i^2 + \sum_{i,j} J_{ij}S_iS_j\big)\Big], \end{aligned}\tag{9}$$

where $\beta = 1/T$ is the inverse temperature and the chemical potential z was introduced to represent the delta function $\delta(N - \sum_i S_i^2)$.

In order to make a connection to the replica field theory (2), we replicate the partition function n times and average over the Gaussian bond distribution. We then take the usual Hubbard-Stratonovich (HS) transformations on the factor $\exp[(\beta^2/4N)\sum_{\alpha,\gamma}(\sum_i S_i^\alpha S_i^\gamma)^2]$ to get the spins on the same site, where the Greek indices denote the replica components. In order to do that, we need to introduce the diagonal $q_{\alpha\alpha}$ and off-diagonal $q_{\alpha\gamma}$ $(\alpha \neq \gamma)$ fields for the corresponding factors. We have

$$\begin{aligned} &\exp\Big[\frac{\beta^2}{4N}\sum_\alpha\Big(\sum_i (S_i^\alpha)^2\Big)^2\Big] = \int\prod_\alpha\big(\frac{N}{4\pi\beta^2}\big)^{\frac{1}{2}} dq_{\alpha\alpha} \\ &\times\exp\Big[-\frac{N}{4\beta^2}\sum_\alpha q_{\alpha\alpha}^2 + \frac{1}{2}\sum_\alpha q_{\alpha\alpha}\sum_i (S_i^\alpha)^2\Big], \end{aligned}\tag{10}$$

and

$$\begin{aligned} &\exp\Big[\frac{\beta^2}{2N}\sum_{\alpha<\gamma}\Big(\sum_i S_i^\alpha S_i^\gamma\Big)^2\Big] = \int\prod_{\alpha<\gamma}\big(\frac{N}{2\pi\beta^2}\big)^{\frac{1}{2}} dq_{\alpha\gamma} \\ &\times\exp\Big[-\frac{N}{2\beta^2}\sum_{\alpha<\gamma} q_{\alpha\gamma}^2 + \sum_{\alpha<\gamma} q_{\alpha\gamma}\sum_i S_i^\alpha S_i^\gamma\Big]. \end{aligned}\tag{11}$$

We can then integrate over the spin variables to obtain the replicated partition function as integrals over the HS fields, $q_{\alpha\alpha}$ and $q_{\alpha\gamma}$ and over the replicated chemical potential z_α:

$$\begin{aligned} \overline{Z^n} &= \int\prod_\alpha\big(\frac{N}{4\pi\beta^2}\big)^{\frac{1}{2}} dq_{\alpha\alpha}\int\prod_{\alpha<\gamma}\big(\frac{N}{2\pi\beta^2}\big)^{\frac{1}{2}} dq_{\alpha\gamma} \\ &\times\int_{-i\infty}^{i\infty}\prod_\alpha \beta\frac{dz_\alpha}{4\pi i}\exp\Big[-\frac{N}{4\beta^2}\sum_\alpha q_{\alpha\alpha}^2 - \frac{N}{2\beta^2}\sum_{\alpha<\gamma} q_{\alpha\gamma}^2 \\ &+\frac{N}{2}\Big(\sum_\alpha \beta z_\alpha - {\rm tr}\ln\big[(\beta z_\alpha - q_{\alpha\alpha})\delta_{\alpha\gamma} - q_{\alpha\gamma}\big]\Big)\Big], \end{aligned}\tag{12}$$

where tr is taken with respect to the replica index.

In the large-N limit, these integrals can be evaluated by the steepest descent method. For $T > T_c \equiv 1$, the saddle points are at $q_{\alpha\gamma} = 0$, $q_{\alpha\alpha} = \beta^2$ and $z_\alpha = \beta + \beta^{-1}$. This is the well-known result [8] from the $N \to \infty$ analysis of the spherical spin glass. We investigate the finite-size corrections in the limit $T \to T_c = 1$ by considering the fluctuations around these saddles. Writing $q_{\alpha\alpha} = \beta^2 + y_\alpha$ and $\beta z_\alpha = 1 + \beta^2 + ix_\alpha$, we have

$$\begin{aligned}\overline{Z^n_{\rm sp}} = C \int \prod_{\alpha<\gamma} \big(\frac{N}{2\pi\beta^2}\big)^{\frac{1}{2}} dq_{\alpha\gamma} \int \prod_{\alpha} \big(\frac{N}{4\pi\beta^2}\big)^{\frac{1}{2}} dy_\alpha \\ \times \int_{-\infty}^{\infty} \prod_\alpha \frac{dx_\alpha}{4\pi} \exp\Big[-\frac{N}{4\beta^2}\Big(\sum_{\alpha,\gamma} q^2_{\alpha\gamma} + \sum_\alpha y^2_\alpha\Big) \\ -\frac{N}{2}\sum_\alpha (y_\alpha - ix_\alpha) \\ -\frac{N}{2}\mathrm{tr}\ln\big[\{1-(y_\alpha - ix_\alpha)\}\delta_{\alpha\gamma} - q_{\alpha\gamma}\big]\Big],\end{aligned} \tag{13}$$

where $C = \exp((nN/2)(1+\beta^2/2))$. If we expand the logarithm in powers of the fields, we find that the quadratic terms in the diagonal fields x_α and y_α inside the exponential are given by

$$-\frac{N}{4}\sum_\alpha \Big[(T^2-1)y^2_\alpha + 2ix_\alpha y_\alpha + x^2_\alpha\Big]. \tag{14}$$

Diagonalizing this quadratic form, we find two eigenvalues with nonvanishing negative real parts at $T = T_c$, which implies that the diagonal fields are hard modes near T_c and can be integrated away without encountering divergences. Therefore the critical behavior is described by the off-diagonal partition function, which can be written as

$$\begin{aligned}Z_{\rm off} \; = \; \int \prod_{\alpha<\gamma} \big(\frac{N}{2\pi\beta^2}\big)^{\frac{1}{2}} dq_{\alpha\gamma} \exp\Big[-\frac{N}{4}(T^2-1)\sum_{\alpha,\gamma} q^2_{\alpha\gamma} \\ +\frac{N}{6}\sum_{\alpha,\beta,\gamma} q_{\alpha\beta}q_{\beta\gamma}q_{\gamma\alpha} + O(q^4)\Big],\end{aligned} \tag{15}$$

where the quartic and higher order terms in $q_{\alpha\gamma}$ all have coefficients proportional to N. A key point of this discussion is to note that, in the limit where $t \equiv (T - T_c)/T_c \to 0$ and $N \to \infty$, the quartic and higher order terms can be neglected if we keep Nt^3 finite. This can be easily seen by rescaling $q_{\alpha\gamma} \to q_{\alpha\gamma}/\sqrt{Nt}$ in (15). In fact we can show that the off-diagonal partition function in this limit is exactly the same as the zero-dimensional cubic field theory defined in (2) and (3) after identifying the expansion parameter as $g^2 = 1/(Nt^3)$ and $\tau = T^2 - 1 = t(2+t) \to 2t$ as $t \to 0$. We have $Z_{\rm off}(N,\beta) \to Z(\frac{1}{Nt^3})$ as $N \to \infty$, $t \to 0$ and $Nt^3 \to$ finite. A similar observation was made for the critical finite size corrections of the Sherrington-Kirkpatrick model in Ref. [13]. (In this paper, the series for the free energy of the toy model was given to order g^4).

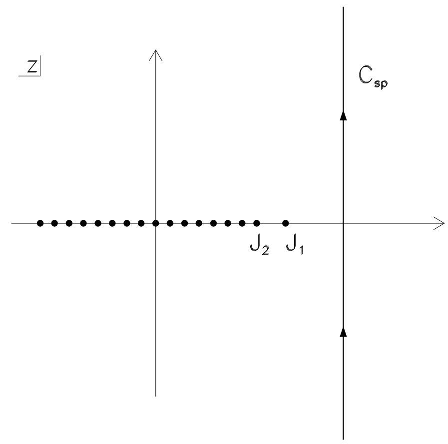

FIG. 3: The integration contour $C_{\rm sp}$ used in Eq. (16). The filled circles represent the eigenvalues J_λ schematically. The largest and the second largest eigenvalues are denoted by J_1 and J_2, respectively.

IV. HIGH-ORDER BEHAVIOR OF THE EXPANSION COEFFICIENTS: TOY PROBLEM

A. Leading-order behavior

Having established the equivalence of the modified spherical spin glass model to the cubic replica field theory, we can analyze the toy problem (2) without using the replicas. Integrating over the spin variables in (9), we obtain

$$Z_{\rm sp} = \beta \int_{-i\infty}^{i\infty} \frac{dz}{4\pi i} \exp\Big[\frac{N\beta z}{2} - \frac{1}{2}\sum_\lambda \ln\Big(\frac{\beta z}{2} - \frac{\beta J_\lambda}{2}\Big)\Big], \tag{16}$$

where J_λ denotes the eigenvalue of the matrix J_{ij}. Note that the contour $C_{\rm sp}$ of integration lies to the right of the largest eigenvalue J_1. (See Fig. 3.)

For large N, the integral is dominated by the saddle point determined by

$$\beta = \frac{1}{N}\sum_\lambda \frac{1}{z - J_\lambda}. \tag{17}$$

In the limit $N \to \infty$, one can evaluate the sum on the right hand side using the Wigner semicircle law for the eigenvalue density $\rho(J_\lambda)$ [8]. Here we are interested in the finite N corrections, in particular, the limit where $N \to \infty$ and $t \to 0$ with Nt^3 held fixed. In this case, one has to find a solution z of (17) for given bond realization J_{ij}, then calculate the free energy $-T \ln Z_{\rm sp}$ from (16), which has to be averaged over the bonds. This is an extremely difficult task to carry out analytically. Instead here we make a self-consistent approximation where we assume the saddle point is located well away (in the sense to be

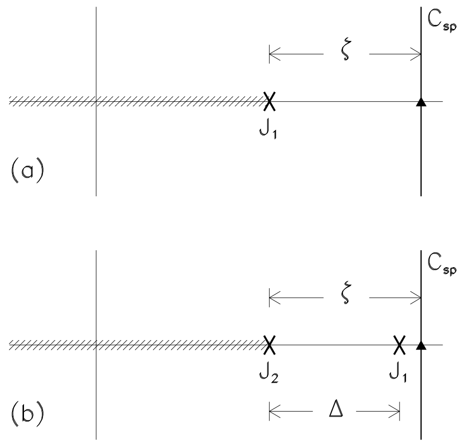

FIG. 4: (a) The disorder distribution responsible for the dominant high-order behavior. The largest eigenvalue J_1 is far away from the saddle point that the contour passes through. The rest of the eigenvalues (the shaded region) are approximated as a continuum distribution. (b) The disorder distribution for the sub-dominant high-order behavior. The separation Δ between the eigenvalues J_1 and J_2 is large so that the eigenvalues smaller than J_2 can be approximated as a continuum.

specified below) from the largest eigenvalue J_1. Since the eigenvalues can be regarded as one-dimensional electric charges interacting logarithmically [14, 15], an analogy to the multipole expansion in electrostatics suggests that we can treat the eigenvalues less than J_1 as a continuous distribution given by the semicircle law. That is to take

$$\rho(J_\lambda) \simeq \frac{2(N-1)}{\pi J_1^2}\sqrt{J_1^2 - J_\lambda^2} + \delta(J_\lambda - J_1). \tag{18}$$

In the limit of $N \to \infty$, $t = (\beta^{-1} - J_1/2)(2/J_1) \to 0$ ($T_c = J_1/2$), and finite $\eta \equiv N^{1/3}t$, we find that the distance of the saddle point from J_1 is scaled as $z - J_1 \sim O(N^{-2/3})$. In terms of $\zeta \equiv N^{2/3}(z - J_1)(2/J_1)$, the saddle point equation (17) reduces to $\eta = \sqrt{\zeta} - 1/\zeta$ in this approximation. As will be shown below, the high-order behavior of the perturbation expansion is described by the small g^2 (or large η) behavior of the partition function. The distance of the saddle point from the largest eigenvalue (measured in terms of ζ) will be large for large η and therefore the approximation can be justified for determining the high-order coefficients of the perturbation expansion. (See Fig. 4 (a).)

Within this approximation, the sum in (16) consists of the term involving J_1 and

$$\frac{1}{N-1}\sum_\lambda{}' \ln(z - J_\lambda) \simeq \frac{z}{J_1^2}\left(z - \sqrt{z^2 - J_1^2}\right) + \ln\left[\frac{1}{2}\left(z + \sqrt{z^2 - J_1^2}\right)\right] - \frac{1}{2}, \tag{19}$$

where the prime indicates the largest eigenvalue is excluded from the sum. By changing the integration variable to ζ in (16) and taking the large-N limit, we obtain

$$\begin{aligned} Z_{\rm sp} &\simeq \frac{\beta e^{-N\beta f_0}}{N^{\frac{1}{3}}}\, e^{\frac{\eta^3}{6}} \int_{-i\infty}^{i\infty} \frac{d\zeta}{4\pi i}\, \frac{\exp(-\frac{1}{2}\eta\zeta + \frac{1}{3}\zeta^{\frac{3}{2}})}{\sqrt{\zeta}} \\ &= \beta N^{-\frac{1}{3}} e^{-N\beta f_0}\, e^{\frac{\eta^3}{6}} \int_C \frac{d\xi}{2\pi i}\, e^{\frac{\xi^3}{3} - \frac{\eta\xi^2}{2}}, \end{aligned} \tag{20}$$

where $\beta f_0 = -\beta J_1/2 + (1/2)\ln(\beta J_1/4) + 1/4 + t^3/6$ and the contour C starts from $|\xi| = \infty$ with $\arg(\xi) = -\pi/4$ and extends to $|\xi| = \infty$ with $\arg(\xi) = \pi/4$. We can evaluate the above integral explicitly as

$$Z_{\rm sp} \simeq \beta N^{-\frac{1}{3}} e^{-N\beta f_0}\, e^{\frac{\eta^3}{12}} \mathrm{Ai}\big(\frac{\eta^2}{4}\big), \tag{21}$$

where Ai is the Airy function. We note that the above expression is valid for both $T > T_c$ ($\eta > 0$) and $T < T_c$ ($\eta < 0$). Since the argument of the Airy function is an even function of η, we obtain, in the large-N limit, $\mathrm{Ai}(\eta^2/4) \sim \exp[-|\eta|^3/12]/\sqrt{2\pi|\eta|}$ using the asymptotic behavior of the Airy function. Therefore, above T_c, the leading contribution to the free energy density, $-N^{-1}\ln Z_{\rm sp}$, in the large-N limit is just βf_0. On the other hand, below T_c, it is given by $\beta f_0 - t^3/6$. One can explicitly check that these quantities coincide with the free energy densities given in Ref. [8] for the spherical model when T approaches T_c from above and below.

We now consider the finite size corrections above T_c. To do this we introduce Z_1 by writing (20) as

$$Z_{\rm sp} = \frac{\beta e^{-N\beta f_0}}{\sqrt{2\pi N t}} Z_1. \tag{22}$$

Note that the square root in the denominator of (22) comes from the Gaussian fluctuations around the large-N saddle point. The finite size corrections relevant to the zero-dimensional cubic field theory comes from Z_1, since we can show that $Z_1 = Z_1(g^2)$ is a function of the expansion parameter $g^2 = \eta^{-3}$ only. It is given by

$$Z_1(g^2) = \sqrt{\frac{2\pi}{g^2}} \int_C \frac{du}{2\pi i}\, \exp\Big[\frac{1}{g^2}\Big(\frac{u^3}{3} - \frac{u^2}{2} + \frac{1}{6}\Big)\Big], \tag{23}$$

where $u = g^{2/3}\xi$. Although $Z_1(g^2)$ can be evaluated analytically as in (21), we calculate the high-order expansion coefficients of Z_1 by an indirect method, where the self-consistency of our approximation is more apparent. (Remember that we are striving to get the behavior of the high-order coefficients exactly; our approximation of the distribution of the eigenvalues as a continuum described by the semicircle law does not give the correct low-order coefficients in the expansion of the free energy). From the integral representation (23), we can analytically continue $Z_1(g^2)$ to any complex g^2 by rotating the contour appropriately. We find that Z_1 has a branch cut along the $g^2 < 0$ axis. The imaginary part of Z_1 is discontinuous crossing this axis. For small $|g|^2$, we can evaluate the discontinuity by the steepest descent method.

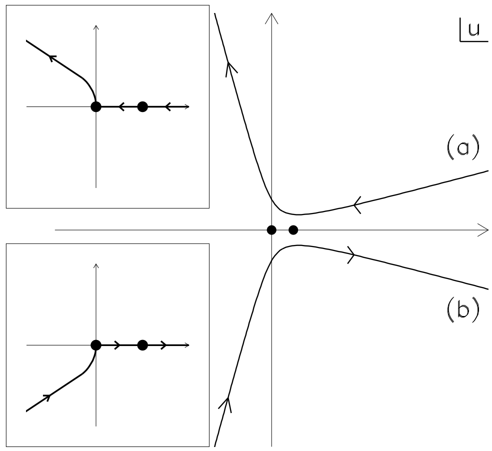

FIG. 5: The rotated contours used in Eq. (23) for (a) $\arg(g^2) = \pi$ and (b) $\arg(g^2) = -\pi$. The insets show how these contours must be deformed to pass through the saddle points $u = 0$ and $u = 1$ (filled circles).

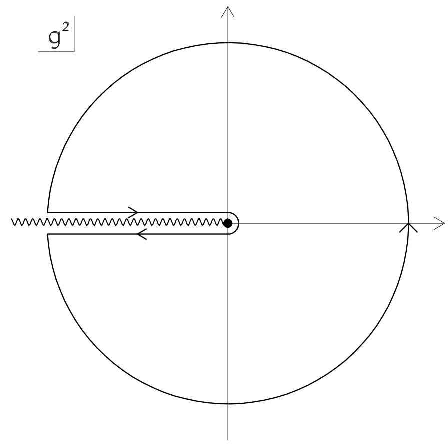

FIG. 6: The integration contour in the g^2 space used to determine the expansion coefficients a_K in Eq. (25). The integral reduces to the one along the branch cut on the $g^2 < 0$ axis as the contribution from the circle vanishes when the radius gets large.

For $\arg(g^2) = \pm\pi$, the contour C is rotated by $\pm\pi/3$ as shown in Fig. 5. Among the two saddle points $u = 0$ and $u = 1$ that C can pass through, the latter produces a real quantity which is the same for $\arg(g^2) = \pm\pi$, while the former is responsible for the discontinuous imaginary part

$$\mathrm{Im}Z_1(g^2; \arg(g^2) = \pm\pi) = \mp\frac{1}{2}\exp(\frac{1}{6g^2})\big[1 + O(g^2)\big]. \tag{24}$$

This can be used to extract the coefficients A_K of the perturbation expansion. We follow the standard procedure [16, 17] by writing a dispersion relation for $Z_1(g^2)$ for $g^2 > 0$ in terms of an integral over a contour that runs around the cut in the negative g^2 axis. (See Fig. 6.) Therefore, the coefficients $a_K \equiv \lim_{n\to 0} A_K/n$ of the perturbation expansion in the toy spin glass field theory is given by

$$a_K \simeq \frac{1}{\pi}\int_{-\infty}^{0} dg^2\, \frac{\mathrm{Im}Z_1(g^2; \arg(g^2) = \pi)}{(g^2)^{K+1}}. \tag{25}$$

For large K, this integral is dominated by the saddle point $g^2 = -1/(6K)$. This implies that the information on $\mathrm{Im}Z_1(g^2)$ for small g^2 can be used to obtain A_K for large K, which justifies the present approximation. We finally obtain

$$a_K \simeq \frac{1}{2\pi}(-6)^K K! K^{-1}\big[1 + O(\frac{1}{K})\big]. \tag{26}$$

The coefficients with alternating signs are consistent with the low-order behavior obtained in Sec. II.

B. Subdominant behavior

There can be other contributions to the free energy from different disorder distributions. For example, when the eigenvalues are distributed in such a way that the saddle point is not very far from the largest eigenvalue, the above approximation breaks down. In this case, we expect a different behavior of the free energy. To handle this, we again make an approximation which can be justified self-consistently. We consider a disorder distribution where the second largest eigenvalue J_2 is well separated from J_1 such that the spectrum below J_2 can be described by the semicircle law. (See Fig. 4 (b).) This assumption is justified in the following analysis. We approximate

$$\rho(J_\lambda) \simeq \frac{2(N-2)}{\pi J_2^2}\sqrt{J_2^2 - J_\lambda^2} + \sum_{i=1,2}\delta(J_\lambda - J_i). \tag{27}$$

and

$$Z_{\rm sp} \simeq \beta\int_{-i\infty}^{i\infty}\frac{dz}{4\pi i}\,\frac{\exp\big[\frac{N\beta z}{2} - \frac{N}{2}\ln(\frac{\beta}{2}) - \frac{1}{2}\sum_\lambda''\ln(z - J_\lambda)\big]}{\sqrt{z - J_1}\sqrt{z - J_2}}, \tag{28}$$

where the double-primed sum excludes $\lambda = 1$ and 2. This sum can be evaluated as in (19) with J_2 replacing J_1. We make the same series of integration variable changes leading to (20) and (23) (using $\zeta \equiv N^{2/3}(z - J_2)(2/J_2)$ and $T_c = J_2/2$ in this case), and take the large-N limit. We obtain $Z_{\rm sp} \simeq \beta e^{-N\beta f_0} Z_2$ where f_0 is the same as before with J_1 replaced by J_2 and

$$Z_2(g^2, \Delta) = \int_C \frac{du}{2\pi i}\,\frac{\exp[\frac{1}{g^2}(\frac{u^3}{3} - \frac{u^2}{2} + \frac{1}{6})]}{\sqrt{u^2 - g^{\frac{4}{3}}\Delta}}, \tag{29}$$

with the eigenvalue spacing $\Delta \equiv N^{2/3}(J_1 - J_2)(2/J_2)$. (Recall $g^{-2} = \eta^3$.)

The contribution from this arrangement of disorder to the free energy, which we denote by $F_{\rm sub}$, is obtained by averaging $-\ln Z_2$ over the distribution $p(\Delta)$ of the eigenvalue spacing Δ. Among the subdominant contributions to the high-order behavior, we focus on those from possible zeros of Z_2. By explicitly evaluating the contour integral (29) numerically for given g, we find that there exists a complex conjugate pair of zeros, Δ_0 and Δ_0^* in the complex-Δ plane. To make analytic progress on the contribution from these zeros, we look at the fluctuation around the saddle point $u = 1$. By writing $u = 1 + igy$ and neglecting $O(g)$ terms, we obtain $Z_2 \simeq (\sqrt{g/2})\widetilde{Z}_2(g^2, \Delta)$, where

$$\widetilde{Z}_2(g^2, \Delta) = \widetilde{Z}_2(v) = \int_{-\infty}^{\infty} \frac{dy}{2\pi} \frac{e^{-y^2/2}}{\sqrt{v + iy}} \tag{30}$$

with $v = (1 - g^{4/3}\Delta)/(2g)$. Note that v is assumed to be of $O(1)$, which means $\Delta \sim O(g^{-4/3})$. This is consistent with the present approximation where the separation of eigenvalues Δ is very large. Writing

$$\frac{1}{\sqrt{v + iy}} = \int_{-\infty}^{\infty} \frac{ds}{\sqrt{2\pi}}\, e^{-s^2(v+iy)}$$

and integrating over y in (30), we have

$$\widetilde{Z}_2 = \int_{-\infty}^{\infty} \frac{ds}{2\pi}\, e^{-vs^2 - s^4/2} = \sqrt{\frac{v}{2}} \frac{e^{v^2/4}}{2\pi} K_{\frac{1}{4}}(\frac{v^2}{4}). \tag{31}$$

The zeros of $\widetilde{Z}_2$ come from the infinitely many zeros of the modified Bessel function $K_{\frac{1}{4}}$ [5, 6]. We can arrange them as complex conjugate pairs, v_m and v_m^*, $m = 0, 1, 2, \ldots$, given approximately for large m by

$$v_m^2 \sim e^{3\pi i/2}\big[-i\ln 2 + (4m + 3)\pi\big] \tag{32}$$

and for $m = 0$ exactly (up to four decimal places) by $v_0^2 = 9.4244 - 0.6928i$. Among the infinitely many zeros, we focus only on the pair v_0 and v_0^* closest to the origin as our numerical evaluation of zeros of (29) is consistent with $\Delta_0 = (1 - 2gv_0)/g^{4/3}$ for small g. The other zeros probably give only subdominant contributions compared to the first ones.

The subdominant contribution to the expansion coefficient denoted by $\widetilde{a}_K$ can be calculated as

$$\widetilde{a}_K = \frac{1}{2\pi i} \oint \frac{dg^2}{(g^2)^{K+1}}\, F_{\rm sub}, \tag{33}$$

where the integral is over a closed contour surrounding the origin. By interchanging the order of integration, we can write

$$\widetilde{a}_K = -\frac{1}{2\pi i} \int_0^{\infty} d\Delta\, p(\Delta) \oint \frac{dg^2}{(g^2)^{K+1}} \ln \widetilde{Z}_2(v). \tag{34}$$

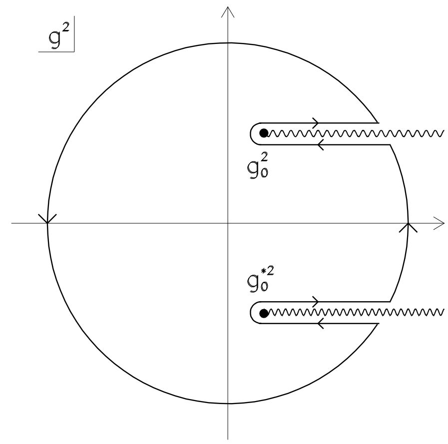

FIG. 7: The deformed contour in g^2 space that leads to Eq. (35). The branch points g_0^2 and g_0^{*2} correspond to the zeros of the logarithm.

The integral over g^2 is done by deforming the contour such that it runs along a circle of radius R and around the branch cuts associated with the zeros, g_0^2 and g_0^{*2} of $\widetilde{Z}_2$ for fixed Δ. (See Fig. 7.) The integral along the circle vanishes as $R \to \infty$ and we have

$$\begin{aligned}\widetilde{a}_K &= \int_0^{\infty} d\Delta\, p(\Delta)\Big[\int_{g_0^2}^{\infty} \frac{dg^2}{(g^2)^{K+1}} + \int_{g_0^{*2}}^{\infty} \frac{dg^2}{(g^2)^{K+1}}\Big] \\ &= \frac{1}{K}\int_0^{\infty} d\Delta\, p(\Delta)\Big[\frac{1}{(g_0^2)^K} + \frac{1}{(g_0^{*2})^K}\Big].\end{aligned} \tag{35}$$

The eigenvalue spacing distribution is known [15] to take the form $p(\Delta) \sim \exp(-(\pi^2/16)(\Delta/D)^2))$ for large Δ when the mean eigenvalue spacing is D. From the numerical results in Ref. [18] on the eigenvalue spacing distribution of real symmetric matrices, we can calculate the mean spacing near the edge of the spectrum as $D \simeq 2.30$. The integral over Δ in (35) can be done using the steepest descent method for large K. The saddle point for the first term in (35) is given by

$$-\frac{\pi^2}{8D^2}\Delta - \frac{2K}{g_0}\Big(\frac{dg_0}{d\Delta}\Big) = 0, \tag{36}$$

where $dg_0/d\Delta$ can be obtained from the defining equation

$$1 - g_0^{\frac{4}{3}}\Delta = 2g_0 v_0. \tag{37}$$

We can solve (36) and (37) for large K to obtain the saddle point values

$$\begin{aligned} g_0^{\rm sad} &= \alpha^{\frac{3}{8}} K^{-\frac{3}{8}}\Big[1 - \Big(\frac{5v_0}{4\alpha^{\frac{5}{8}}}\Big)K^{-\frac{3}{8}} + O(K^{-\frac{3}{4}})\Big], \\ \Delta^{\rm sad} &= \alpha^{-\frac{1}{2}} K^{\frac{1}{2}}\Big[1 - 2\big(1 - \frac{5}{6\alpha}\big)v_0 \alpha^{\frac{3}{8}} K^{-\frac{3}{8}} + O(K^{-\frac{3}{4}})\Big], \end{aligned}$$

where $\alpha = \pi^2/(12D^2) \simeq 0.155$. Note that the large-$K$ behavior corresponds to small g and large Δ with $\Delta \sim g^{-4/3}$, which means that these results are entirely in the regime of the present approximation. Inserting these into (35) we finally obtain

$$\begin{aligned}\tilde{a}_K \;\sim\; & (K!)^{\frac{3}{4}}\, a^K \exp\left[-bK^{\frac{5}{8}} + O(K^{\frac{1}{4}})\right] \\ & \times \cos\left(cK^{\frac{5}{8}} + O(K^{\frac{1}{4}})\right), \end{aligned} \tag{38}$$

where $a = \alpha^{-3/4} \simeq 4.05$, $b = -3\alpha^{3/8}\mathrm{Re}(v_0) \simeq 3.15$ and $c = 3\alpha^{3/8}\mathrm{Im}(v_0) \simeq 3.38$. Compared with (26), this is a subdominant contribution containing only a fractional power of $K!$. The coefficients do not alternate in sign as in (26) but oscillate with a cosine function with an increasing periodicity. The situation is similar to that in the zero-dimensional disordered ferromagnets [4, 5, 6], where this type of oscillation also occurs in the subdominant terms. We expect that as in the case of disordered ferromagnets the subdominant terms make the resummation of the series non-trivial such that a straightforward Borel summation is spoiled. However, it seems likely that the series could be resummed in other ways and anyway, the evidence from Ref. [4] suggests that the straightforward Padé-Borel method works well for short series even in the presence of subdominant terms.

There are obviously other types of bond distribution which could give rise to subdominant contributions to the high-order behavior of the perturbation series besides the one studied in this subsection. We suspect that the type studied here provides the largest of these contributions but we have no proof of this.

V. REPLICA APPROACH TO HIGH ORDER BEHAVIOR IN THE TOY PROBLEM

While the mapping of the toy integral, Eq. (2), to the spherical model has enabled us to obtain direct estimates for the high-order terms of its perturbative expansion, in order to obtain high-order estimates for the d-dimensional field theory and hence for critical exponents we have to discover how to obtain the same high-order estimates directly from the integral in the replica variables $q_{\alpha\beta}$. Once this has been done the extension to field theory is relatively straightforward and is carried out in the next Section. Unfortunately the direct replica approach is neither obvious nor rigorous. Without the results obtained from the mapping to the spherical model we would have had no confidence in the replica procedure which we were forced to use.

It is useful to first examine the integral for the special case of $n = 3$ when the integrals can be done explicitly and exist if w is pure imaginary. This case was analyzed in Ref. [7]. Here we follow the same analysis to clarify some points which will be important in the case of general n. For $n = 3$, we have after setting $\tau = 1$ for simplicity

$$Z_3(w) = \frac{4\pi}{(2\pi)^{3/2}} \int_0^\infty dR R^2 e^{-\frac{R^2}{2}} f(wR^3), \tag{39}$$

where

$$\begin{aligned} f(wR^3) \;&=\; \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta e^{wR^3 \sin^2\theta \cos\theta \cos\phi \sin\phi} \\ &=\; \frac{\pi}{3} I_{\frac{1}{6}}\left(\frac{wR^3}{6\sqrt{3}}\right) I_{-\frac{1}{6}}\left(\frac{wR^3}{6\sqrt{3}}\right) \end{aligned} \tag{40}$$

with the modified Bessel function $I_\nu(x)$. For pure imaginary coupling $w = iw'$, w' real, $f = (\pi/3)J_{1/6}(w'R^3/6\sqrt{3})J_{-1/6}(w'R^3/6\sqrt{3})$ with the ordinary Bessel function $J_\nu(x)$. We can easily see that the integral in (39) is well-defined and real in this case.

When analytically continued to real coupling w, Z_3 develops an imaginary part which is discontinuous crossing the branch cut along the real w axis. For small values of coupling, one can calculate the discontinuity using the steepest descent method on the integral in (39). Since the main contribution to the imaginary part comes from the saddle point $R \sim \mathrm{O}(1/w)$, we first study the asymptotic behavior of f when $|wR^3|$ is very large, which is

$$f(wR^3) \sim \frac{\sqrt{3}}{wR^3}\left[\exp\left(\frac{wR^3}{3\sqrt{3}}\right) - \exp\left(-\frac{wR^3}{3\sqrt{3}}\right) + \sqrt{3}i\right]. \tag{41}$$

The above expansion is valid for $0 \le \arg(w) \le \pi$, or for $0 \le \arg(w^2) \le 2\pi$. For $\arg(w) = 0$, only the first exponential in (41) is important and the integral in (39) is dominated by the saddle points $R_1 = 0$ and $R_2 = \sqrt{3}/w$. The steepest descent direction at R_2 is perpendicular to the real-R axis along which the integral generates the imaginary part. We can deform the contour in (39) such that it starts from R_1 along the positive real-R axis toward R_2 and makes an upward turn at R_2. (Examples of similar deformations are in Ref. [19].) We note that, since we only pass a half of the steepest descent path of R_2 in this way, the Gaussian integral coming from the fluctuation around R_2 produces a half of the total fluctuation contribution. Keeping this in mind, we evaluate the integral to obtain $\mathrm{Im}Z_3(w) = \exp(-\frac{1}{2w^2})$ for $\arg(w) = 0$. For $\arg(w) = \pi$, the saddle points are $R_1' = 0$ and $R_2' = -\sqrt{3}/w$ and the continuation of the contour used for $\arg(w) = 0$ to this case is the one which makes the downward turn at R_2'. We finally obtain the discontinuity in the imaginary part along the branch cut on the real-w axis as

$$\mathrm{Im}Z_3(w) = \pm\exp(-\frac{1}{2w^2}), \tag{42}$$

where the positive and negative signs correspond to $\arg(w^2) = 0$ and $\arg(w^2) = 2\pi$, respectively. This exponentially small imaginary contribution to the partition function at the physical coupling can be used to determine the large order behavior of the perturbation expansion, but at the same time its presence indicates that the

cubic field theory for $n = 3$ is ill-defined and requires stabilizing quartic terms for its existence.

We now study the case of general n. For small values of the expansion parameter, $w^2/(\tau/2)^3$, we can evaluate the integrals in Eq. (2) using the steepest descent method. Saddle points are found by solving

$$-\tau q_{\alpha\beta} + w \sum_{\gamma} q_{\alpha\gamma} q_{\gamma\beta} = 0. \tag{43}$$

The trivial solution of this equation $q_{\alpha\beta} = 0$ is the starting point of the perturbative expansion. Non-perturbative terms arise from its non-trivial solution. The set of saddles which we study are $q_{\alpha\beta} = q$, when both α and β lie in the interval between 1 and r, where and $q = \tau/(r-2)w$. This can be described schematically as

$$q_{\alpha\beta} = \begin{pmatrix} \boxed{q} & 0 \\ 0 & 0 \end{pmatrix}_{\underbrace{\quad}_{r}\underbrace{\quad}_{n-r}}. \tag{44}$$

Clearly other types of saddle exist besides this. We first focus on this class of saddles, which we call scheme I. The Hamiltonian density for this saddle is given by

$$\mathcal{H}_s = \frac{r(r-1)\tau^3}{12(r-2)^2 w^2}. \tag{45}$$

There are more solutions to Eq. (43) with the same Hamiltonian (45), which can be obtained by switching the signs of some of $q_{\alpha\beta}$. One can take, for example, $q_{1\beta} = -q$ for $\beta \neq 1$, and $q_{\alpha\beta} = q$ for $\alpha, \beta \neq 1$. Since one can pick any other subscript than 1, the number of such solutions is r. One can also switch the signs of two different sets such as $q_{1\gamma} = q_{2\gamma} = -q$ for $\gamma \neq 1, 2$ and keep all the other elements $q_{\alpha\beta} = q$. The number of such solutions is $r(r-1)/2$. Similarly, one can switch the signs of $3, 4, \ldots, r-1$ different sets of $q_{\alpha\beta}$. However, we can show that the solutions obtained by switching the signs of k different sets are equivalent to those from picking $r-k$ different sets for $k = 1, 2, \ldots, r-1$. Therefore, the total number $S(r)$ of the solutions with the Hamiltonian (45) is

$$S(r) = \sum_{k=0}^{[r/2]} {}^{r}C_k = \begin{cases} 2^{r-1}, & r \quad \text{odd}, \\ 2^{r-1} + \frac{r!}{2[(r/2)!]^2}, & r \quad \text{even}, \end{cases} \tag{46}$$

where $[x]$ denotes the largest integer less than or equal to x.

We need to include the Gaussian fluctuations around these saddles. The matrix of the second derivatives has six distinct eigenvalues: a "breather" mode with eigenvalue $-\tau$ which is non-degenerate, one other negative eigenvalue, $-\tau/(r-2)$ which is $(n-r)$-fold degenerate, and eigenvalues $2\tau/(r-2)$, $(r-1)$-fold degenerate, $r\tau/(r-2)$, $(r(r-3)/2)$-fold degenerate, $(2r-3)\tau/(r-2)$, $(r-1)(n-r)$-fold degenerate, and τ, $(n-r)(n-r-1)/2$-fold degenerate. Setting $\tau = 1$ for simplicity and collecting the contributions from the saddle points to Gaussian order, we obtain

$$\begin{aligned} Z^{(\mathrm{I})}(w^2) &= [1 + \mathrm{O}(w^2)] \\ &+ \frac{1}{2} \sum_{r=3}^{n} {}^{n}C_r \, S(r) \exp\left[-\frac{r(r-1)}{12(r-2)^2 w^2} \right] (-1)^{\frac{1}{2}} \\ &\times \left(\frac{r-2}{2} \right)^{\frac{r-1}{2}} \left(\frac{r-2}{r} \right)^{\frac{r(r-3)}{4}} \left(-(r-2) \right)^{\frac{n-r}{2}} \\ &\times \left(\frac{r-2}{2r-3} \right)^{\frac{(n-r)(r-1)}{2}} (1)^{\frac{(n-r)(n-r-1)}{4}} [1 + \mathrm{O}(w^2)], \end{aligned} \tag{47}$$

where the first term corresponds to the usual perturbation expansion in w. The sum over r starts at 3 since in Eq. (43) the index γ must differ from both α and β. The factor ${}^{n}C_r$ denotes the number of ways of introducing r non-zero blocks. Collecting the contributions from the saddles in this way, instead of deforming a contour in a multi-dimensional complex space, determines the partition function up to an overall factor. The analysis of the $n = 3$ case suggests that there is an overall factor of 1/2 coming from the fact that, for the nontrivial saddles, only a half of the Gaussian integrals contribute compared to the perturbative one. Indeed, for $n = 3$, one can explicitly check that the non-perturbative part of (47) reduces to Eq. (42). The negative eigenvalues are responsible for the factor $(-1)^{(n-r+1)/2}$, which generates the imaginary part in Z.

For finite n, the saddles in the scheme I correspond to the partition function which is well defined except for real w resulting in a branch cut on the positive w^2 axis. The discontinuity of the imaginary part across the cut is exponentially small $\sim \exp(-n(n-1)/(12(n-2)^2 w^2))$. (The imaginary part of (47) is dominated by the $r = n$ term, since the Hamiltonian (45) decreases monotonically as r increases.) If the cubic spin glass field theory is well-defined for real coupling w, we expect that the cut moves to the negative w^2 axis as we take the $n \to 0$ limit and that there is an exponentially small discontinuity across the negative w^2 axis. The migration of the cut as the analytic continuation of $n \to 0$ is taken is exactly what happens in the percolation problem [7]. In that case, the Hamiltonian $\mathcal{H}_s$ for the saddles depends explicitly on n such that it changes sign as $n \to 0$. In the present case, $\mathcal{H}_s$ in (45) is independent of n, and there is no way of producing an exponentially small discontinuity across the negative w^2 axis from these saddles. Therefore, we conclude that the saddles in the scheme I are not sufficient to describe the partition function in the $n \to 0$ limit.

This observation leads us to consider another type of solution of Eq. (43), which we call scheme II. It is inspired by the replica symmetry breaking scheme used by two of us [20] to describe the free energy fluctuations. Instead of taking all $q_{\alpha\beta}$ nonzero for $\alpha, \beta = 1, 2 \ldots r$ as in the

previous scheme, we set $q_{\alpha\beta} = q$ only when α and β belong to the m blocks of size r/m on the diagonal of the $r \times r$ matrix as

$$q_{\alpha\beta} \;=\; \left(\begin{array}{cccc|c} \boxed{q} & & & & \\ & \boxed{q} & & & \\ & & \ddots & & \\ & & & \boxed{q} & \\ \hline & & & & \end{array}\right) . \qquad (48)$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{r}\;\underbrace{\qquad\quad}_{n-r}$$

The key point of the construction of these saddles is that we let $m \to \infty$ before $n \to 0$. For finite m, this scheme is just a generalization of the scheme I which produces only subleading terms. For $m \to \infty$, however, the Hamiltonian becomes

$$\mathcal{H}_s = \frac{m(\frac{r}{m})(\frac{r}{m}-1)\tau^3}{12(\frac{r}{m}-2)^2 w^2} \to -\frac{r\tau^3}{48w^2}, \qquad (49)$$

which has the opposite sign to the one in the scheme I. This can now describe the partition function where the cut lies on the imaginary w axis. The solution to (43) is $q = \tau/(r/m-2)w \to -\tau/2w$ as $m \to \infty$.

The matrix of the second derivatives necessary to include the Gaussian fluctuations around the saddles, has two distinct positive eigenvalues, $\tau/2$ and τ, which are, respectively, $r(n-r)$-fold and $(n-r)(n-r-1)/2$-fold degenerate. There exist a null eigenvalue, $r(r-3)/2$-fold degenerate, and one negative eigenvalue $-\tau$, r-fold degenerate. This negative eigenvalue is responsible for the factor $(-\tau)^{-\frac{r}{2}}$ which generates the imaginary part for odd values of r. The total number of solutions that can be obtained by switching the sign of $q_{\alpha\beta}$ is $[S(r/m)]^m \to 2^{r/2}$ as $m \to \infty$. (This limit exists if we assume r/m is even.) From (49), we can see that the saddles with smallest r dominate for pure imaginary w. Therefore, the leading contribution to the imaginary part of (50) should come from the first i.e. $r = 1$ term as we can see no reason why it should be excluded in this kind of replica symmetry breaking scheme. Thus

$$\begin{aligned} \mathrm{Im} Z^{(\mathrm{II})}(w^2) \;&=\; \pm \frac{{}^nC_1}{2} \exp\left[\frac{\tau^3}{48w^2}\right] 2^{\frac{1}{2}} \left(\frac{2}{\tau}\right)^{\frac{1}{2}(n-1)} \\ &\quad\times \left(\frac{1}{\tau}\right)^{\frac{1}{4}(n-1)(n-2)+\frac{1}{2}} [1+O(w^2)], \end{aligned} \qquad (50)$$

where the upper and lower signs correspond to $\arg(w) = -\pi/2$ and $\arg(w) = \pi/2$, respectively.

We take the $n \to 0$ limit of (50) using the fact ${}^nC_r = n(-1)^{r-1}/r + O(n^2)$, which can be derived from $1/\Gamma(n-r+1) = \Gamma(r-n)\sin(\pi(r-n))/\pi$. We finally obtain the discontinuity of the imaginary part of the partition function (2) in the limit $n \to 0$ across the cut on the imaginary w ($w^2 < 0$) axis as

$$\lim_{n\to 0} \mathrm{Im}\frac{Z^{(\mathrm{II})}(w^2)}{n} \sim \pm \exp\left(\frac{\tau^3}{48w^2}\right). \qquad (51)$$

We cannot obtain the precise prefactor to the exponential term because we have neglected the contributions of the massless eigenvalue. In principle, this soft mode could be integrated out by identifying the underlying symmetries associated with the saddle point. This does not seem obvious to us. However, its contribution is subdominant to that from the exponential and we shall ignore its contribution.

The discontinuity in the imaginary part of the partition function can be used to extract the coefficients A_K of the perturbation expansion. To do that we write the above result in terms of the expansion parameter g^2. Recalling $\tau = 2t$, we can write the right hand side of Eq. (51) as $\pm\exp(1/6g^2)$ for $\arg(g^2) = \mp\pi$. This is the same as Eq. (24) up to the undetermined prefactor. Therefore we obtain exactly the same high-order behavior as in (25)

$$\lim_{n\to 0} \frac{A_K}{n} \sim (-6)^K K!. \qquad (52)$$

Without the spherical model mapping one would have had reservations about the likely correctness of the replica procedure used.

VI. HIGH ORDER TERMS OF THE ϵ EXPANSION

The starting point for obtaining the large order form of the ϵ expansion is the Hamiltonian of Eq. (1) but without the quartic terms:

$$\mathcal{H} = \frac{1}{4}\sum_{\alpha,\beta}(\nabla q_{\alpha\beta})^2 + \frac{\tau}{4}\sum_{\alpha,\beta} q_{\alpha\beta}^2 - \frac{w}{6}\sum_{\alpha,\beta,\gamma} q_{\alpha\beta}q_{\beta\gamma}q_{\gamma\alpha}. \qquad (53)$$

Our treatment closely follows that of McKane [7]. The saddle points which are the analogue of Eq. (43) of the toy replica calculation are the instantons which satisfy the equation

$$\nabla^2 q_{\alpha\beta} = -\tau q_{\alpha\beta} + w\sum_{\gamma} q_{\alpha\gamma}q_{\gamma\beta}. \qquad (54)$$

For evaluating the high-order coefficients in the ϵ expansion we can set $\tau = 0$ and look for a solution of the form $q_{\alpha\beta} = w^{-1}d_{\alpha\beta}\phi_c(r)$ in $d = 6$ dimensions. Such a solution, which decouples replica indices from the spatial dependence r, exists if

$$d_{\alpha\beta} = \sum_{\gamma} d_{\alpha\gamma}d_{\gamma\beta}, \qquad (55)$$

and

$$\nabla^2\phi_c(\mathbf{r}) = \phi_c^2(\mathbf{r}). \qquad (56)$$

There are spherically symmetric solutions of Eq. (56):

$$\phi_c(r) = -\frac{24\lambda^2}{[\lambda^2 r^2+1]^2}, \qquad (57)$$

where the parameter λ reflects the dilatation invariance of Eq. (56). We shall take for $d_{\alpha\beta}$ the replica symmetry broken solution of scheme II. Then the energy of the instanton is

$$E = \int d^6\mathbf{r} \left[\frac{1}{4}\sum_{\alpha,\beta}\left(\nabla\phi_c(r)\right)^2 d_{\alpha\beta}^2/w^2 - \frac{w}{6}\sum_{\alpha,\beta,\gamma} d_{\alpha\beta}d_{\beta\gamma}d_{\gamma\alpha}\phi_c^3(r)/w^3\right]. \quad (58)$$

Using the result that

$$\int d^6\mathbf{r}\,\left(\nabla\phi_c(r)\right)^2 = -\int d^6\mathbf{r}\,\phi_c^3(r) = \frac{1152\pi^3}{5}, \quad (59)$$

the energy of the instanton is

$$E = \left(\frac{1152\pi^3}{5}\right)\frac{1}{48w^2} = \frac{3}{40g_R^2} \quad (60)$$

where $g_R^2 = K_6 w^2$ and $K_6 = S_6/(2\pi)^6$ and $S_6 = \pi^3$ is the surface area of a six dimensional sphere of unit radius.

The leading terms in the large order behavior of the ϵ expansion are obtained by replacing g_R^2 by its fixed point value, which to lowest order is $\epsilon/2$ [12] and by the usual saddle-point arguments the coefficient of ϵ^K for large K for any critical exponent goes like

$$\sim K!\left(-\frac{20}{3}\right)^K \quad (61)$$

The next most dominant term is a factor of the form K^b. The value of b depends on the critical exponent being studied and is beyond the scope of this paper. To determine its value a treatment is needed of the massless modes which arise in the Gaussian fluctuations around the instanton solution.

Inspection of the first three terms in the ϵ expansion for η and $\nu^{-1} - 2 + \eta$ show that these terms are not growing anything like as rapidly as predicted at large K. It is perhaps not surprising, therefore, that a Padé-Borel analysis of the series does not yield good numerical values for the critical exponents in three dimensions.

VII. DISCUSSION

In summary we have studied the nature of the perturbation expansion of the zero-dimensional cubic replica field theory of spin glasses. By mapping this to the problem of critical finite-size corrections in a modified spherical spin glasses, we have determined the high-order behavior of the perturbation expansion coefficients. To the leading order, the coefficients alternate in sign, but there is a subleading contribution where the terms in the perturbation series show a cosine-like oscillation. In practice, the effects of these sub-dominant terms will be small, making a simple Padé-Borel resummation of the series useful, as was found to be the case in a similar situation for the disordered ferromagnet [4].

Non-perturbative terms are also present in spin glasses. These are Griffiths singularities and arise from regions where the values of the couplings J_{ij} produce a smaller amount of frustration and hence a locally enhanced transition temperature. A discussion of their form has been given in Ref. [21]. Similar singularities exist for disordered ferromagnets and it is widely believed that their effects are very small. To our knowledge no quantitative discussion of these singularities has been made for spin glasses and their study remains to be done. (The toy problem, because it is zero-dimensional, is free of Griffiths singularities).

The ϵ expansion for the critical exponents gives disappointing results as regards applications to real spin glasses. This is not just due to the fact that $\epsilon = 3$ in three dimensions as in Ref. [11] good results were obtained for the exponents of the percolation problem in three dimensions from an ϵ expansion with the same number of terms. We do not understand the origin of this problem.

However, to our mind, the most significant remaining problem is what motivated this entire study. Namely, does perturbation theory (i.e. the loop expansion) work well in the spin glass phase or does the existence of "droplets" in finite dimensional spin glasses indicate that it fails completely? Our work does indicate though that perturbation theory is useful in the high-temperature phase.

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